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Simple algorithm for factorized dynamics of the $\mathfrak{g}_n$-automaton

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Abstract
We present an elementary algorithm for the dynamics of recently introduced soliton cellular automata associated with quantum affine algebra $U_q(\mathfrak{g}_n)$ at $q = 0$. For $\mathfrak{g}_n = A_n^{(1)}$, the rule reproduces the ball-moving algorithm in Takahashi–Satsuma's box–ball system. For non-exceptional $\mathfrak{g}_n$ other than $A_n^{(1)}$, it is described as a motion of particles and anti-particles which undergo pair-annihilation and creation through a neutral bound state. The algorithm is formulated without using representation theory or crystal basis theory.

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1. Introduction

The box–ball system invented by Takahashi and Satsuma [TS, T] is a remarkable example of one-dimensional soliton cellular automata. It represents the dynamics of balls hopping along the array of boxes under a certain rule. The evolution equations have been studied extensively in [TTMS, TTM, TNS] by means of the ultradiscretization of soliton equations.

In [HKT1, FOY, HHIKTT], the box–ball system was identified with a solvable vertex model in statistical mechanics [B] in a low-temperature limit. Mathematically it brought a crystal basis theory [K] into the game, and led to a generalization labelled with affine Lie algebras $\mathfrak{g}_n$. We call the resulting system the $\mathfrak{g}_n$-automaton. See [HKT1] for the construction for non-exceptional $\mathfrak{g}_n$, and [HKOTY] for the study of soliton scattering. The box–ball system corresponds to the $\mathfrak{g}_n = A_n^{(1)}$ case [FOY, HHIKTT] (see also [HIK]).

The dynamics of the $\mathfrak{g}_n$-automaton is governed by the combinatorial R [KMN] in crystal theory, which specifies the local interaction of a box and a carrier in the language of the box–ball system. Except for the $A_n^{(1)}$ case [NY], the complexity of the combinatorial R [HKOT1,HKOT2] makes it tedious to compute the dynamics of the associated $\mathfrak{g}_n$-automaton. Fortunately the difficulty was overcome in [HKT2] for the automaton having a carrier with infinite capacity, where a factorization of the time evolution into simple Weyl group reflections was proved.

Table 1.

$\mathfrak{g}_n$	B	$(j_d, \ldots, j_1)$
$A_n^{(1)}$	$\{1, 2, \ldots, n+1\}$	$(2, 3, \ldots, n+1)$
$A_{2n-1}^{(2)}$	$\{1, 2, \ldots, n, -n, \ldots, -2, -1\}$	$(2, 3, \ldots, n, -1, -n, \ldots, -3, -2)$
$A_{2n}^{(2)}$	$\{1, 2, \ldots, n, -n, \ldots, -2, -1, \emptyset\}$	$(2, 3, \ldots, n, -1, -n, \ldots, -3, -2, \emptyset)$
$B_n^{(1)}$	$\{1, 2, \ldots, n, 0, -n, \ldots, -2, -1\}$	$(2, 3, \ldots, n, 0, -n, \ldots, -3, -2)$
$C_n^{(1)}$	$\{1, 2, \ldots, n, -n, \ldots, -2, -1\}$	$(2, 3, \ldots, n, -1, -n, \ldots, -3, -2, -1)$
$D_n^{(1)}$	$\{1, 2, \ldots, n, -n, \ldots, -2, -1\}$	$(2, 3, \ldots, n, -n, \ldots, -3, -2)$
$D_{n+1}^{(2)}$	$\{1, 2, \ldots, n, 0, -n, \ldots, -2, -1, \emptyset\}$	$(2, 3, \ldots, n, 0, -n, \ldots, -3, -2, \emptyset)$

The aim of this paper is to translate the factorized dynamics [HKT2] stated in the crystal language into an analogue of the ball-moving algorithm, and thereby to describe the $\mathfrak{g}_n$-automaton in terms of a simple evolution rule. For simplicity we shall restrict ourselves to the case where all the boxes have minimal capacity. In the crystal formulation of the time evolution: $B_M \otimes (\cdots \otimes B_{m_i} \otimes B_{m_{i+1}} \otimes \cdots) \simeq (\cdots \otimes B_{m_i} \otimes B_{m_{i+1}} \otimes \cdots) \otimes B_M$, the situation corresponds to the choice $\forall m_i = 1$ and $M = \infty$. The factorized dynamics for the $A_n^{(1)}$-automaton reproduces the decomposition of the ball-moving algorithm in the box–ball system into a finer procedure where one only touches the balls with a fixed colour. For the other $\mathfrak{g}_n$ in question, the dynamics is decomposed similarly, where each finer procedure (denoted by K_j in section 2) is now described as the motion of particles and anti-particles with a fixed colour exhibiting pair-annihilation and creation through a neutral bound state.

In section 2 we specify the set of local states and give the algorithm. In section 3 we present examples. In section 4 we list the patterns that behave as solitons. In section 5 we briefly explain the relation of the algorithm in section 2 and the result in [HKT2]. The key is the formula (5).

Although our notations originate in the crystal theory, the contents in sections 2, 3 use no result from it. Despite being just a translation of a more general result in [HKT2], we hope this paper possesses its own role in making the $\mathfrak{g}_n$-automaton more accessible and familiar to the people working on cellular automata and discrete integrable systems.

2. Algorithm

We formulate the $\mathfrak{g}_n$-automaton associated with the non-exceptional affine Lie algebra $\mathfrak{g}_n = A_n^{(1)}(n \geqslant 1), A_{2n-1}^{(2)}(n \geqslant 3), A_{2n}^{(2)}(n \geqslant 2), B_n^{(1)}(n \geqslant 3), C_n^{(1)}(n \geqslant 2), D_n^{(1)}(n \geqslant 4)$ and $D_{n+1}^{(2)}(n \geqslant 2)$. Actually the algorithm itself given below is meaningful formally for $A_3^{(2)}, B_2^{(1)}, D_2^{(1)}$ and $D_3^{(1)}$ as well. First we specify the set B of local states and the sequence $j_1, \ldots, j_d \in B$ as given in table 1.

Here $\emptyset$ should be understood as a symbol and not an empty set. By the definition $d = n$ for $A_n^{(1)}$, $d = 2n-2$ for $D_n^{(1)}$, $d = 2n-1$ for $A_{2n-1}^{(2)}, B_n^{(1)}$, $d = 2n$ for $A_{2n}^{(2)}, C_n^{(1)}$ and $D_{n+1}^{(2)}$.

We let

$$W = \{(\ldots, b_i, b_{i+1}, \ldots) \in \cdots \times B \times B \times \cdots \mid b_i = 1 \text{ for } |i| \gg 1\} \tag{1}$$

be the set of states of the automaton at a fixed time. Setting

$$J = \{j_1, \ldots, j_d\}$$

one finds

$$J = \begin{cases} B \setminus \{1, -1\} & \text{if} \quad \mathfrak{g}_n = B_n^{(1)}, D_n^{(1)}, D_{n+1}^{(2)} \\ B \setminus \{1\} & \text{otherwise.} \end{cases}$$

For $j \in J$ (hence $j \neq 1$), we define

$$j^* = \begin{cases} j & \text{if} \quad j \in \{0, -1, \emptyset\} \\ -j & \text{if} \quad j \in J \setminus \{0, -1, \emptyset\}. \end{cases} \tag{2}$$

The time evolution of the $\mathfrak{g}_n$-automaton $T : W \to W$ takes the factorized form

$$T = K_{j_d} \ldots K_{j_2} K_{j_1}. \tag{3}$$

To define the operator $K_j : W \to W$ $(j \in J)$, one regards an element $(\ldots, b_i, b_{i+1}, \ldots)$ of W as the one-dimensional array of boxes containing b_i in the ith box. (The $(i + 1)$th box to the right of the ith box.) In what follows we regard the box containing $1 \in B$ as an empty box. Thus, taking some $b\,(\neq 1) \in B$ away from a box means the change of the local states $b \to 1$. Similarly, putting $b\,(\neq 1) \in B$ into an empty box means the change $1 \to b$. Note that the boundary condition in (1) says that only finitely many boxes are non-empty. Under this convention the operator K_j is defined by the following algorithm which consists of four steps:

Step 1. Replace each -1 by a pair j, j^* within a box.

Step 2. Pick the leftmost j (if any) and move it to the nearest right box which is empty or containing just j^*. (Boxes involving the pair j, j^* are prohibited as the destination.)

Step 3. Repeat step 2 for those j that are not yet moved until all the j are moved once.

Step 4. Replace the pair j, j^* within a box (if any) by -1.

Remark 1. For $\mathfrak{g}_n = A^{(1)}_n$ where no -1 is present in B and J, steps 1 and 4 become void and the above procedure shrinks to the well known ball-moving algorithm in the box–ball system [T].

Remark 2. When $j \in \{-1, 0, \emptyset\}$, one can have two j within a box due to (2). As for those duplicated j, it does not matter which is left or right, but of course they should be distinguished according to whether they are moved or not yet moved. See $K_\emptyset$ in example 3.

Remark 3. A little inspection shows that K_{-1} actually admits a simpler description which is essentially the ball-moving algorithm in the box–ball system:

Step 1′. Pick the leftmost -1 (if any) and move it to the nearest right empty box.

Step 2′. Apply step 1′ for those -1 that are not yet moved (if any).

Step 3′. Repeat step 2′ until all the -1 are moved once.

Remark 4. Consider the $\mathfrak{g}_n \neq A^{(1)}_n$ case and interpret the local states as given in table 2.

Since the set $B \setminus \{1, -1\}$ is invariant under the interchange $j \leftrightarrow j^*$, the state j^* may be viewed as the anti-particle of j and vice versa. In this sense the bound state -1 is neutral, and especially 0 and $\emptyset$ represent the 'self-neutral' particles that can still form a bound state with another one. Under this interpretation, the above algorithm for K_j describes the motion of right-moving particles of colour j seeking an empty box or a free partner, i.e. an anti-particle j^* not yet paired with another j, to form a neutral bound state within a box.

Table 2.

B	Entry in the box
1	Empty
$j \neq \pm 1$	Particle of colour j
-1	Bound state of j and j^*

The algorithm for K_j can also be stated in terms of local rules, which we shall now explain. For $j \in J$ we introduce a map $L_j : (\mathbb{Z}_{\geqslant 0}) \times B \to B \times (\mathbb{Z}_{\geqslant 0})$ as follows. Let the diagram $\begin{array}{ccc} & b & \\ l & \xrightarrow{j} & l' \\ & b' & \end{array}$ denote $L_j : (l, b) \mapsto (b', l')$ (j is attached to the horizontal line).

(1) $j \notin \{1, 0, -1, \emptyset\}$ case. Assume $l \in \mathbb{Z}_{\geq 0}$, $b \in B \backslash \{j, -j, 1, -1\}$.

$$\begin{array}{ccc} & j & \\ l & \xrightarrow{j} & l+1 \\ & 1 & \end{array} \quad \begin{array}{ccc} & -j & \\ l+1 & \xrightarrow{j} & l \\ & -1 & \end{array} \quad \begin{array}{ccc} & -j & \\ 0 & \xrightarrow{j} & 0 \\ & -j & \end{array} \quad \begin{array}{ccc} & -1 & \\ l & \xrightarrow{j} & l+1 \\ & -j & \end{array} \quad \begin{array}{ccc} & 1 & \\ l+1 & \xrightarrow{j} & l \\ & j & \end{array} \quad \begin{array}{ccc} & 1 & \\ 0 & \xrightarrow{j} & 0 \\ & 1 & \end{array} \quad \begin{array}{ccc} & b & \\ l & \xrightarrow{j} & l \\ & b & \end{array}$$

(2) $j \in \{0, \emptyset\}$ case. Assume $l \in \mathbb{Z}_{\geq 0}$, $b \in B \backslash \{j, 1, -1\}$.

$$\begin{array}{ccc} & -1 & \\ l & \xrightarrow{j} & l+2 \\ & 1 & \end{array} \quad \begin{array}{ccc} & j & \\ 0 & \xrightarrow{j} & 1 \\ & 1 & \end{array} \quad \begin{array}{ccc} & j & \\ l+1 & \xrightarrow{j} & l+1 \\ & j & \end{array} \quad \begin{array}{ccc} & 1 & \\ l+2 & \xrightarrow{j} & l \\ & -1 & \end{array} \quad \begin{array}{ccc} & 1 & \\ 1 & \xrightarrow{j} & 0 \\ & j & \end{array} \quad \begin{array}{ccc} & 1 & \\ 0 & \xrightarrow{j} & 0 \\ & 1 & \end{array} \quad \begin{array}{ccc} & b & \\ l & \xrightarrow{j} & l \\ & b & \end{array}$$

(3) $j = -1$ case. Assume $l \in \mathbb{Z}_{\geq 0}$, $b \in B \backslash \{1, j\}$.

$$\begin{array}{ccc} & j & \\ l & \xrightarrow{j} & l+1 \\ & 1 & \end{array} \quad \begin{array}{ccc} & 1 & \\ l+1 & \xrightarrow{j} & l \\ & j & \end{array} \quad \begin{array}{ccc} & 1 & \\ 0 & \xrightarrow{j} & 0 \\ & 1 & \end{array} \quad \begin{array}{ccc} & b & \\ l & \xrightarrow{j} & l \\ & b & \end{array}$$

For $\mathfrak{g}_n = A_n^{(1)}$ we only have the third case with the j understood as $j \in \{2, 3, \ldots, n+1\}$, and the resulting presentation of K_j first appeared in [HIK].

Given an automaton state $(\ldots, b_i, b_{i+1}, \ldots) \in W$, there exists an integer m such that $b_{m'} = 1$ for all $m' < m$ owing to the boundary condition (1). Fix any such m. Then the operator $K_j : W \to W$ maps $(\ldots, b_i, b_{i+1}, \ldots)$ to $(\ldots, c_i, c_{i+1}, \ldots)$, where $c_{m'} = b_{m'}$ for all $m' < m$. The remaining $c_m, c_{m+1}, \ldots$ are determined by the composition of L_j as

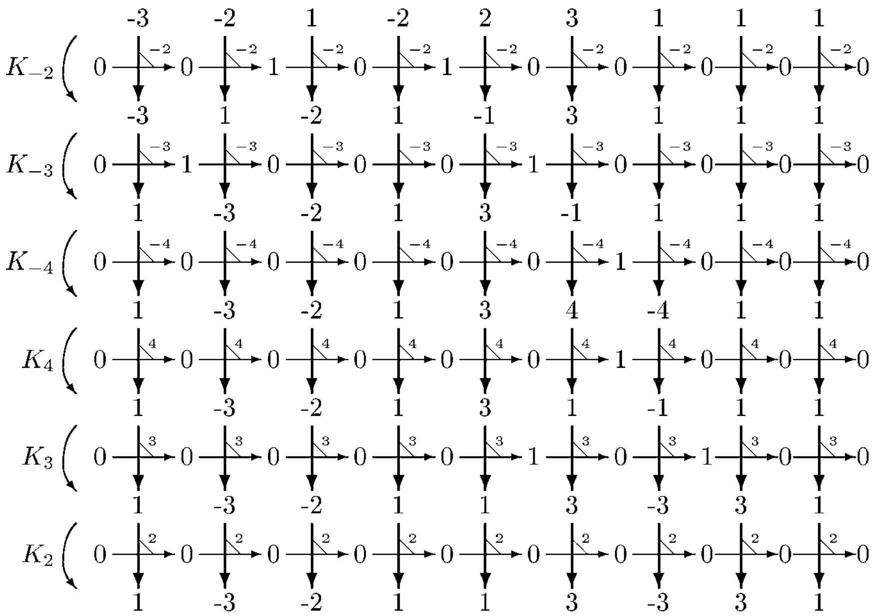
$$\begin{array}{ccccccc} & b_m & & b_{m+1} & & \cdots & \\ 0 & \xrightarrow{j} & & \xrightarrow{j} & & \xrightarrow{j} & \\ & c_m & & c_{m+1} & & \cdots & \end{array} \tag{4}$$

It is easy to see that the result is independent of the choice of m. The non-negative integers l on the horizontal line stand for the number of colour j particles on the carrier. The diagrams for L_j are viewed as the loading and unloading processes of the colour j particles when the carrier proceeds to the right. They match the algorithm stated before. Perhaps the above formulation of K_j using L_j is easier to program.

3. Examples

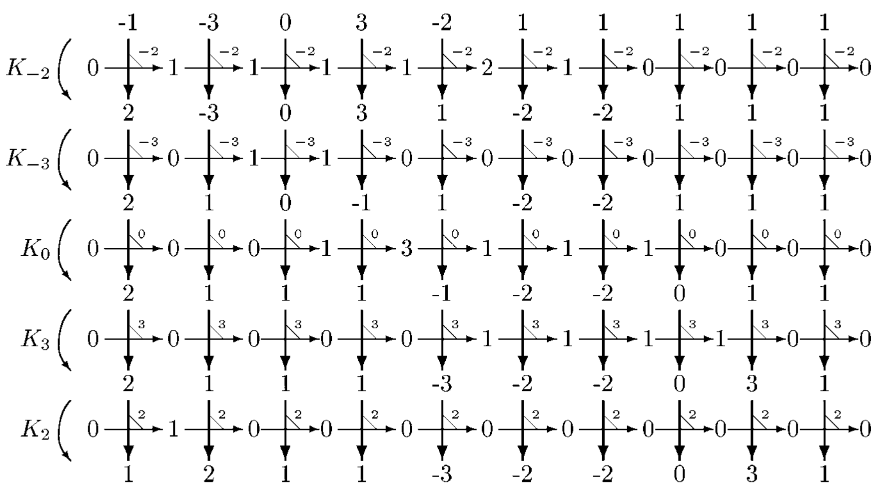
Let us present examples of the factorized dynamics. We suppress the trivial left tail in the time evolution and only depict the part that corresponds to the composition of (4) vertically.

Example 1. $\mathfrak{g}_n = D_4^{(1)}$. $T = K_2 K_3 K_4 K_{-4} K_{-3} K_{-2}$.



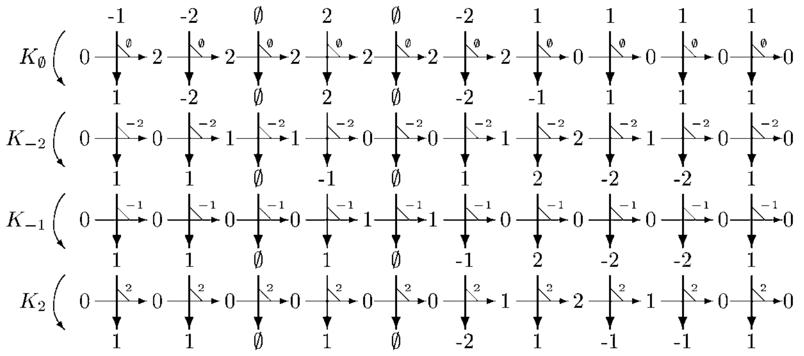
$$T : (-3, -2, 1, -2, 2, 3, 1, 1, 1, \ldots) \mapsto (1, -3, -2, 1, 1, 3, -3, 3, 1, \ldots).$$

Example 2. $\mathfrak{g}_n = B_3^{(1)}$. $T = K_2 K_3 K_0 K_{-3} K_{-2}$.



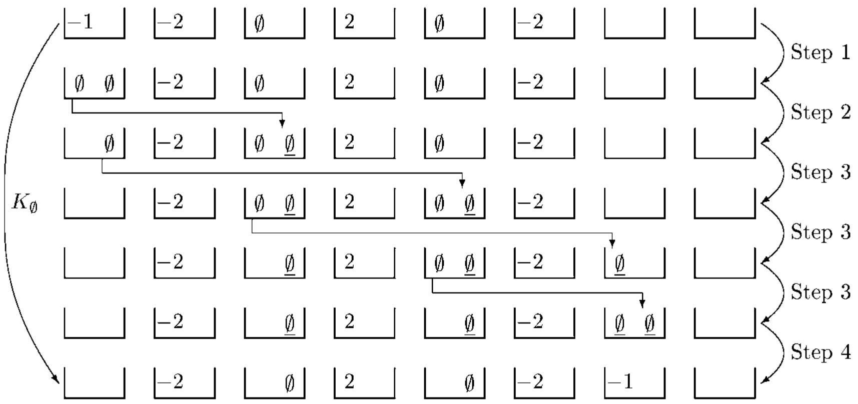
$$T : (-1, -3, 0, 3, -2, 1, 1, 1, 1, 1, \ldots) \mapsto (1, 2, 1, 1, -3, -2, -2, 0, 3, 1, \ldots).$$

Example 3. $\mathfrak{g}_n = A_4^{(2)}$. $T = K_2 K_{-1} K_{-2} K_{\emptyset}$.



$$T : (-1, -2, \emptyset, 2, \emptyset, -2, 1, 1, 1, 1, \ldots) \mapsto (1, 1, \emptyset, 1, \emptyset, -2, 1, -1, -1, 1, \ldots).$$

Along the steps 1–4 in Section 2, the action of $K_\emptyset$ in the above goes as follows.



As cautioned in remark 2, one must distinguish the moved $\emptyset$ and those not yet moved. Here we have marked the moved ones with underlines.

4. Solitons

To save the space we shall write, for example, $2^{y_2}(-3)^{y_{-3}}$ to signify the configuration of the local states $\overbrace{2, \ldots, 2}^{y_2}, \overbrace{-3, \ldots, -3}^{y_{-3}}$ in a segment of an automaton state for non-negative integers y_2 and y_{-3}, etc. In what follows we assume $y_b \in \mathbb{Z}_{\geqslant 0}$ for any $b \in B$. For each $\mathfrak{g}_n$ consider the following configurations and define v from them:

$A_n^{(1)}$: $(n+1)^{y_{n+1}} \ \ldots \ 3^{y_3} \ 2^{y_2}$

$$v = \sum_{i=2}^{n+1} y_i$$

$A_{2n-1}^{(2)}$: $(-2)^{y_{-2}} \ (-3)^{y_{-3}} \ \ldots \ (-n)^{y_{-n}} \ n^{y_n} \ \ldots \ 3^{y_3} \ 2^{y_2}$

$$v = \sum_{i=2}^{n} (y_i + y_{-i})$$

$$A^{(2)}_{2n} : (\emptyset)^{y_\emptyset}(-1)^{y_{-1}}\ (-2)^{y_{-2}}\ \ldots\ (-n)^{y_{-n}}\ n^{y_n}\ \ldots\ 2^{y_2}\ 1^{y_1} \qquad y_\emptyset \in \{0,1\} \quad y_1 = y_{-1}$$

$$v = y_\emptyset + \sum_{i=1}^{n}(y_i + y_{-i})$$

$$B^{(1)}_{n} : (-2)^{y_{-2}}\ (-3)^{y_{-3}}\ \ldots\ (-n)^{y_{-n}}\ 0^{y_0}\ n^{y_n}\ \ldots\ 3^{y_3}\ 2^{y_2} \qquad y_0 \in \{0,1\}$$

$$v = y_0 + \sum_{i=2}^{n}(y_i + y_{-i})$$

$$C^{(1)}_{n} : (-1)^{y_{-1}}\ (-2)^{y_{-2}}\ \ldots\ (-n)^{y_{-n}}\ n^{y_n}\ \ldots\ 2^{y_2}\ 1^{y_1} \qquad y_1 = y_{-1}$$

$$v = \sum_{i=1}^{n}(y_i + y_{-i})$$

$$D^{(1)}_{n} : (-2)^{y_{-2}}\ (-3)^{y_{-3}}\ \ldots\ (-n)^{y_{-n}}\ n^{y_n}\ \ldots\ 3^{y_3}\ 2^{y_2} \qquad y_n y_{-n} = 0$$

$$v = \sum_{i=2}^{n}(y_i + y_{-i})$$

$$D^{(2)}_{n+1} : (\emptyset)^{y_\emptyset}(-1)^{y_{-1}}\ (-2)^{y_{-2}}\ \ldots\ (-n)^{y_{-n}}\ 0^{y_0}\ n^{y_n}\ \ldots\ 2^{y_2}\ 1^{y_1} \qquad y_\emptyset, y_0 \in \{0,1\} \quad y_1 = y_{-1}$$

$$v = y_\emptyset + y_0 + \sum_{i=1}^{n}(y_i + y_{-i}).$$

In an element of W, the above configuration is called a soliton with amplitude v if it is surrounded by sufficiently many $1 \in B$. (For an amplitude v soliton, 1^v in its right suffices.) The data $\{y_b\}$ are the internal labels of a soliton. Under the time evolution T (3), the amplitude v solitons propagate stably to the right with velocity v if they stay sufficiently away from configurations different from $1 \in B$ [HKT1]. It is a good exercise to check this claim by using the algorithm for T in section 2. Moreover, the following facts have been proved concerning the collisions of solitons with distinct amplitudes [HKOTY]:

(1) The set of amplitudes remains invariant under the collisions.
(2) The two-soliton scattering rule is characterized by the combinatorial R of $U_q(\mathfrak{g}_{n-1})$.
(3) Multi-soliton scattering factorizes into the two-soliton ones.

Leaving the precise statements to [HKOTY], we here include an example of the collision of two solitons.

Example 4. $\mathfrak{g}_n = C^{(1)}_3$. To make the space uniform, we denote $-j$ by $\overline{j}$ for $j = 1, 2, 3$. Then the evolution of a two-soliton state under T^t $(0 \leqslant t \leqslant 5)$ is depicted as follows:

$$\begin{aligned}
t = 0 &: \cdots 11\bar{1}\bar{2}\bar{2}311111\bar{2}\bar{3}2111111111111111111111111 \cdots \\
t = 1 &: \cdots 1111111\bar{1}\bar{2}\bar{2}3111\bar{2}\bar{3}2111111111111111111111 \cdots \\
t = 2 &: \cdots 111111111111\bar{1}\bar{2}311\bar{2}\bar{2}\bar{3}211111111111111111 \cdots \\
t = 3 &: \cdots 1111111111111111\bar{1}3111\bar{2}\bar{2}\bar{2}\bar{3}2111111111111 \cdots \\
t = 4 &: \cdots 1111111111111111111\bar{1}311111\bar{2}\bar{2}\bar{2}\bar{3}21111111 \cdots \\
t = 5 &: \cdots 1111111111111111111111\bar{1}31111111\bar{2}\bar{2}\bar{2}\bar{3}211 \cdots
\end{aligned}$$

One observes that the initial two solitons $\bar{1}\bar{2}\bar{2}31$ and $\bar{2}\bar{3}2$ are scattered into the final solitons $\bar{1}31$ and $\bar{2}\bar{2}\bar{2}\bar{3}2$ with a phase shift. In this way, the two-body scattering rule consists of the exchange of the internal labels of solitons and the phase shift.

Table 3.

$\mathfrak{g}_n$	$i_{k+d}, \ldots, i_{k+1}$
$A_n^{(1)}$	$2, 3, \ldots, n, n+1$
$A_{2n-1}^{(2)}, B_n^{(1)}$	$0, 2, 3, \ldots, n-1, n, n-1, \ldots, 3, 2, 0$
$A_{2n}^{(2)}, C_n^{(1)}, D_{n+1}^{(2)}$	$1, 2, \ldots, n-1, n, n-1, \ldots, 2, 1, 0$
$D_n^{(1)}$	$0, 2, 3, \ldots, n-2, n-1, n, n-2, n-3, \ldots, 3, 2, 0$

5. Relation to crystal theory

The factorized dynamics in this paper is a translation of a result in [HKT2]. By regarding the local states $-1, \ldots, -n$ here as $\overline{1}, \ldots, \overline{n}$, B can be identified with the crystal B_1 as a set. In equation (33) of [HKT2], take the integer k as $k = n-1$ for $D_n^{(1)}$ and $k = n$ for the other $\mathfrak{g}_n$. It leads to $a_k = 1$ (cf tables I, II and equation (11) therein) for all the $\mathfrak{g}_n$ in question, which implies the boundary condition $\ldots 111 \ldots$. The time evolution in this case is given by $T = \mathcal{T}_{k+d}$, where the data $i_{k+d}, \ldots, i_{k+1}$ in equation (33) there read as given in table 3.

Under the convention $\mathcal{T}_k = \mathrm{id}$, the operator K_j in this paper has emerged from the formula

$$K_{j_r} = \mathcal{T}_{k+r}(\mathcal{T}_{k+r-1})^{-1} \qquad 1 \leqslant r \leqslant d. \tag{5}$$

We have verified that the right-hand side yields the composition of L_{j_r} described in section 2 by an explicit calculation. Similar factorized dynamics can be formulated corresponding to other boundary conditions than $1 \in B$ in (1) by choosing a different k in the above. However, such variations do not affect the qualitative feature of the automata.

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